MATH 3060 Assignment 5 solution

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1. $$T^n x(t) = 1 + \frac{t^2}{2} + \frac{1}{2} \cdot \left(\frac{t^2}{2}\right)^2 + \cdots + \frac{1}{n!} \cdot \left(\frac{t^2}{2}\right)^n.$$

   It converges to the limit
   $$\exp\left(\frac{t^2}{2}\right).$$

2. We will apply Theorem 3.4 in lecture 12. Take $\Phi = \mathrm{I} + \Psi = -\cos x + 2x^4 + x$, $x_0 = 0$ and $y_0 = -1$.
   $$\begin{aligned} |\Psi(x) - \Psi(x')| &= |\sin \xi + 8\xi^3||x - x'| \\ &= 10r|x - x'| \end{aligned}$$
   when $|x|, |x'| < r < 1$. To make sure -1.001 is in the image, we need $10r < 1$ and $(1 - 10r)r > 0.01$. This is satisfied, for example when $r = 1/20$.

3. We will apply Theorem 3.4 in lecture 12. Take $\Phi = \mathrm{I}+\Psi$, $x_0 = y_0 = (0, 0)$, and
   $$\Phi(x, y) = (\sin x - 2y^4, \sin y + x^2).$$
   When $||(x - x', y - y')|| < r < 1$ we have,
   $$\begin{aligned} ||\Psi(x, y) - \Psi(x', y')||^2 &= ((\cos \xi_1 - 1)(x - x') - 8\zeta_1^3(y - y'))^2 + (2\xi_2(x - x') + (\cos \zeta - 1)(y - y'))^2 \\ &\le (r(x - x') + 8r(y - y'))^2 + (2r(x - x') + r(y - y'))^2 \end{aligned}$$
   $\le 90r^2||(x - x', y - y')||^2.$

   We want $\sqrt{90}r < 1$ and $(1-\sqrt{90}r)r > 0.01$ this time. This can be achieved by setting $r = 1/20$.

4. Define
   $$Ty(x) = g(x) + \lambda \int_0^1 K(x, t)y(t))dt.$$

Then

$$\begin{aligned}|Ty_1(x) - Ty_2(x)| &\leq \lambda \int_0^1 |K(x,t)||y_1(t) - y_2(t)| \\ &\leq \lambda M d_\infty(y_1, y_2).\end{aligned}$$

Where $M = \sup K$. If we choose λ so that $\lambda M < 1$, then T ia a contraction in $(C[0,1], d_\infty)$, and hence has a unique fixed point.